Challenges in Mentoring Mathematical Biology Model Construction: Quantification and Context

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*The researcher observed a group of undergraduate students and faculty mentors collaborating in the development of a model for a student chosen topic in epidemiology. Results suggest that both students and mentors struggled with key understandings necessary to develop the model. Students struggled with conceiving of their compartment model as relating quantities, and mentors struggled with tracking and attending to the biological constraints of the problem the students chose.*

*Keywords:* Ordinary differential equations, mathematical modeling, model construction, student-led projects, quantification

Choosing quantities and their relationships is a critical part of successfully approaching a mathematical model. Equations and graphs in word problem and modeling contexts represent relationships between quantities and their rates of change. However, the notion of quantity receives little focus in the teaching school mathematics (Smith & Thompson, 2007; Thompson, 2008, 2011). Thompson describes quantification as the process of conceptualizing of an attribute of an object as having a measure. Understanding how students and mathematicians imagine and interpret quantities is critical to understanding the process of model development.

Research in mathematical modeling and quantification typically focuses on students working pre-chosen tasks (Bliss et. al., 2006; Gravemijer, 1994; Goldin, 1997; Lesh & Doerr, 2003; Thompson, 2011; Steffe and Thompson 2000). Sometimes these tasks are quite open, and students go through cycles of model development; however, in assigning a task to students, there are constraints placed on students as part of intentionally guiding the students' conceptual development (ibid). It has been argued in the past that these constraints limit students' experiences in developing their own research questions (Castillo-Garsow, 2014; Castillo-Garsow & Castillo-Chavez, 2015). Furthermore, pre-chosen tasks place teachers or teacher-researchers in a dominant role where the teacher has an opportunity to learn and know everything there is to know about the problem. So these pre-chosen tasks rarely highlight difficulties in modeling or mentoring modeling that a teacher would experience.

For example, Camacho et al. (2003) found that choosing one's own project and research question creates situations in which students take the lead in researching topics far outside a mentors' area of expertise, essentially reducing the mentor to a role of consultant rather than leader. Mentors provide mathematical expertise, guiding students by suggesting appropriate tools and techniques. However, students take the lead in these projects, providing the background that defines the problem from subject area research and personal experience (Castillo-Garsow & Castillo-Chavez, 2015). These role reversals in which students are the experts are important sources of motivation and self-efficacy. Choosing a topic that mentors know less about allow students to participate and contribute in ways other than mathematical performance (Rubel, 2017), which is an area in which students cannot compete with mentors (Camacho et. al 2003; Castillo-Garsow & Castillo-Chavez, 2015).

This perspective creates a dichotomy of two ways in which difficulties with quantification can occur: understanding the background context but having difficulty mathematizing, or having an expert understanding of the mathematical tools for modeling, but having difficulty understanding and connecting those tools to the background context due to inexperience with the context itself. Quantification research typically focuses on the former, but rarely focuses on the latter. This study focuses on both. As students and mentors interacted with each other, this paper identifies challenges that arose in quantification for both students and mentors as they collaborated to construct a model in a student-led project.

## Methods

This study occurred in the fifth week of an eight-week summer REU in mathematical biology. Prior to this study, the students had taken a three-and-a-half-week course consisting of lecture, computer lab work, and textbook exercises in dynamical systems. Following this course work, students self-recruited into groups of three to five, and chose a topic of interest. During the fifth week, students made daily presentations on their topic to a panel of faculty and graduate mentors who provided feedback. In the final three weeks of the program, students completed the analysis of their model and wrote a technical report on their project. Four groups of students chose to participate in the study, and this a case study from one of those groups. Analysis of the other groups can be found elsewhere (Castillo-Garsow, 2021, 2022).

The group of students in this study was formed of five undergraduate students who chose to construct a model for controlling a disease that can be treated but not cured. The students working on this project eventually developed their work into a published journal article, the citation for which is omitted for privacy. The students made six presentations over six days to a panel of faculty and graduate mentors who provided feedback to the students. Each proposal presentation was video and audio recorded, and the audio recordings were transcribed. Transcripts were open coded (Corbin & Strauss, 2014), and from that coding, themes emerged that identified and explained the primary areas of conflict between mentors and students. The results here are a case study of those transcripts, focusing on creating a narrative of those conflicts (Flyvbjerg, 2006). The purpose of this case study is to identify challenges than mentors and students may encounter while collaborating on a student-led applied mathematics project, both to inform mentors and to inspire future research.

## Results

The groups' research question was focusing on the cost effectiveness of treating individuals with mild symptoms of a disease, compared to the current practice of only treating patients in the severe symptom stage. These mild symptoms occurred in many different diseases, meaning that treating individuals with mild symptoms would result in treating many individuals who did not have the disease of interest with medication for the wrong disease. The students proposed studying this question with a system of ordinary differential equations (ODE model).

### Student Challenge: Quantity vs. Category

In the students' first attempt at constructing a disease model (Figure 1), the students classified individuals only by their symptoms. The category $I_1$ therefore contained both individuals who had the disease of study, and individuals who had the same symptoms of a difference disease. Students imagined that individuals who did not have the disease of study would return to S, while individuals who did have the disease of study would progress to $I_2$ or L. Describing this model, a student said: "After presenting mild symptoms, those mild symptoms go away, then they go back to the susceptible class. Only those who have [the disease] proceed to a progression to the severe symptoms, which are only for [the disease]."

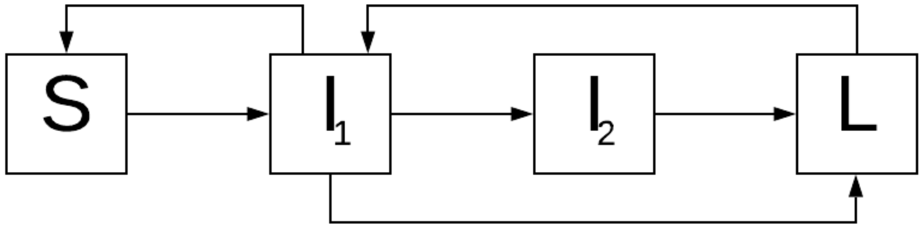


*Figure 1: A simplified flow diagram of the disease group's first model. Box S represents susceptibles, $I_1$ represents individuals with mild disease symptoms from several diseases.. $I_2$ represents individuals with severe symptoms unique to the disease of study. L represents asymptomatic individuals.*

However, in an ODE model, these variables only track a number of individuals. $I_1$ would necessarily be a numerical quantity of a number of individuals with mild symptoms, meaning that information about who does or does not really have the disease of study could not be stored in this information structure. Students may have made this mistake because they were imagining tracking individuals moving through the categories S, $I_1$, $I_2$, and L; rather than imagining S, $I_1$, $I_2$, and L as simple numbers of people. In other words, the students were not conceiving of S, $I_1$, $I_2$, and L as the values of quantities that could be measured (Thompson, 2011).

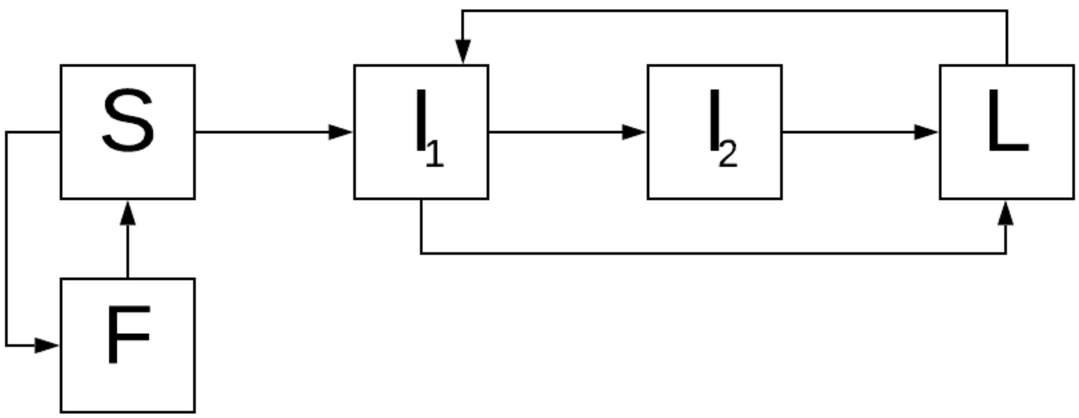


*Figure 2. Students' second model, showing an F compartment for individuals falsely diagnosed with the disease of study. The students' fourth and fifth models also had a similar compartment forming a closed loop with S.*

Mentors provided feedback on this model informing them that they needed a separate compartment for individuals who did not have the disease "I think you're going to need a separate class for those people" (day 3) and "There's no way to do this without a separate compartment" (day 3). But at this time, mentors did not explain that tracking individuals was not possible in an ODE, or that S, $I_1$, $I_2$, and L were numbers. Students responded by creating a compartment F for individuals falsely diagnosed with the disease of study (Figure 2), but this was changed again in the third version (Figure 3).

In the students' third version of the model (Figure 3), they repeated their categorization mistake with a new compartment. Here students imagined that all individuals exhibiting symptoms would receive the same treatment, so all treated individuals were placed in a single compartment. Again, the students imagined that from this compartment, individuals who did not have the disease of study would return to S, and individuals who did have the disease would advance to L, and that the model would somehow keep track of which individuals were which. This resulted in a model with a path from susceptible to asymptomatically infected passing only through treatment – implying that it was possible for treatment itself to cause infection. This time, mentors addressed problems with tracking individuals. As one mentor put it:

> You can't do that because if a person who doesn't have the illness and a person who does have the illness, and they go to the same thing. What you're doing after that is you're saying, both people who don't have it and do have it can now become [asymptomatic]. (Mentor, day 5)

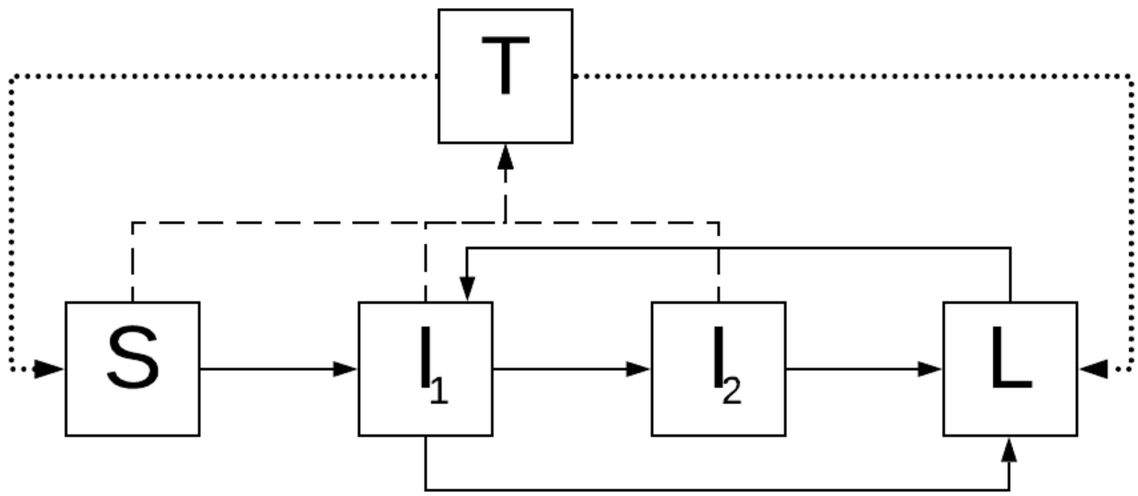


*Figure 3. A simplified flow diagram of the disease group's third model, with an added T compartment for treated individuals. Dashed lines represent individuals moving into treatment. Dotted lines represent individuals moving out of treatment.*

Another perspective here is that students may have been adopting the point of view of a physician-observer, rather than the point of view of the disease itself. Students wanted to lump all the mild symptoms into a single category because all mild symptoms look alike. Similarly, they wanted to lump all treated individuals together because they all were receiving the same treatment. However, from the point of view of the disease, individuals in these categories had very different diseases, and therefore needed to be tracked separately. In either case, $I_1$ and T did not represent numerical quantities.

**Mentoring Challenge: Dynamics over the research question.**

Because students were tracking the cost-effectiveness of treatments that had a risk of being wasted on individuals with another disease, the students needed a way to track the number of falsely diagnosed individuals. These falsely diagnosed individuals would add to the cost without controlling the disease. The students included a compartment, F, for this in their second, fourth, and fifth models (Figure 2).

This F compartment was isolated from the rest of the model in that the variable F did not appear in equations for $I_1$, $I_2$, or L. Because this compartment did not affect disease dynamics, mentors objected to its inclusion as unnecessary, and frequently forgot that the compartment was needed to answer the research question of cost. Examples include: "If you only consider dynamics, F compartment does nothing" (day 3) and "You asked the question how to treat early treatment for [the disease]. That part [F] has nothing to do with [the disease]. Why do you have to include this here?" (day 6).

**Mentoring Challenge: Testing vs. diagnosis**

In the disease model, a key component of the cost was the risk of treating other diseases. This risk was increased because no available test that could distinguish mild symptoms of this disease from mild symptoms of other diseases. The tests that did exist could only be used during severe symptoms, when testing was unnecessary because the symptoms were characteristic. The students frequently stated that there was no test, or that testing was only possible in the $I_2$ severe stage. However, mentors frequently assumed that there was testing or screening occurring in $I_1$. See the following excerpt from day 3:

*Student*: The current test that we have now, there is no way to test if you have [the] disease. There's absolutely no way. The only way you test it if you go here [$I_2$] and you have a lesion here to take samples
*Mentor*: The cost of this testing, and the patient, the $I_1$. The same test?
*Student*: There's no testing for $I_1$ in that.

Confusion about testing continued through day 5, where mentors continued to ask questions and suggest changes to the model that involved "testing" or "screening" individuals in $I_1$. For example, on day 5, suggesting changes to the third model (Figure 3) by incorporating screening: "So then you screen that [S] and once you screen that you put it here [$I_1$], [you] do not go here [T]." At least some of the confusion arose from students frequently referring to falsely diagnosed individuals as a "false positive," suggesting the presence of a test.

## Discussion

Previous research from the project showed that mentoring had the most impact on students' decisions when the mentors focused on asking questions about the biological background, and making suggestions about the mathematics (Castillo-Garsow, 2021). That result is consistent with perspectives found in literature on these student-led projects, which describe students as having topic context expertise, while mentors have mathematical expertise (Camacho et al, 2013; Castillo-Garsow & Castillo-Chavez, 2015). Effective mentoring of a student-led project involves respecting the respective expertise of both students and mentors. This project shows an alternate perspective on the same phenomenon. Here, the challenges in model development arose from students struggling to adopt a mathematical perspective on the problem, while mentors struggled to understand the biological context.

The students in this project had a strong understanding of the biology, but had difficulty communicating that understanding to the mentors. The mentors had difficulty with setting aside their preconceived ideas of what the biology of this disease would be and imagined that testing and treatment occurred in ways that they were more familiar with. Mentors also struggled to attend to contextual concerns – such as cost – over mathematical concerns, such as the dynamical behavior of the model itself.

The students also struggled with quantification. They imagined the modeling process as the story of individuals moving through categories, and/or the perspectives of observers of those individuals. However, writing a mathematical model requires imagining not just individuals, but also numbers of individuals. The students' repeated difficulties in making the transition from category to quantity resulted in errors in the base structure of the model and the corresponding mathematical equations. Mentors initially responded by only correcting the surface level mistake. It was only after the mistake was repeated in a new way that mentors addressed the foundations of ODE model construction with students, specifically the principle that individuals did not have histories that could be tracked through compartments. Here the necessary mentoring expertise was in mathematics in understanding the assumptions and limitations of an ODE model, but also in pedagogical content knowledge by forming a model of the students' interpretation of the model and addressing individual tracking.

## Conclusion

The results here add to the literature on student-led projects in mathematical biology modeling. The results of this paper suggest that the reversal of roles described in previous literature (Camacho et al. 2013; Castillo-Garsow & Castillo-Chavez, 2015) is not only sufficient for a successful project (Castillo-Garsow, 2021), but also necessary. In this example, difficulties in model construction came from participants operating outside of their areas of expertise. Students struggled with mathematical concepts, and mentors struggled with biological concepts. However, this struggle outside of ones area of expertise should not be avoided. Rather it was mutual teaching between students and mentors that enabled the participants to collaborate and develop a successful model that was eventually published.

The results of this research also suggest that more research is needed in the ways that mathematicians come to understand or struggle to understand scientific concepts as part of mathematical quantification. Quantification research cannot only address the mathematization of well understood scientific contexts, but must also explore how a developing understanding of the context influences the conceptualization of quantities and the development of quantitative relationships. In particular, further study of mathematics experts developing models of unfamiliar scientific problems would greatly add to our understanding of quantification.

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